Dynamics of generalizations of the AGM continued fraction of Ramanujan. Part I: divergence.

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version 1.00—July 19, 2004

Abstract

We study several generalizations of the AGM continued fraction of Ramanujan inspired by a series of recent articles in which the validity of the AGM relation and the domain of convergence of the continued fraction were determined for certain complex parameters [2, 3, 4]. A study of the AGM continued fraction is equivalent to an analysis of the convergence of certain difference equations and the stability of dynamical systems. Using the matrix analytical tools developed in [4], we determine the convergence properties of deterministic, and stochastic difference equations and so divergence of their corresponding continued fractions.

1 Introduction

For the sequence $a := (a_n)_{n=1}^{\infty}$, denote the continued fraction $\mathcal{S}_1(a)$ by

$$\mathcal{S}_1(a) = \cfrac{1^2a_1^2}{1+\cfrac{2^2a_2^2}{1+\cfrac{3^2a_3^2}{1+\ddots}}} \tag{1.1}$$

We study the convergence properties of this continued fraction for deterministic and random sequences (a_n). For the deterministic case we derive our most general results from an examination of periodic sequences, that is, sequences satisfying $a_j = a_{j+c}$ for all j and some finite c. In this

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case we will sometimes represent the sequence only by its base cycle $a = (a_1, a_2, \ldots, a_c)$. Our definition of $\mathcal{S}_1$ leads to a slight idiosyncrasy with respect to other definitions for the case $c = 2$ since these begin the continued fraction with the *second* element of the sequence. Many special cases of the above continued fraction for particular choices of a have been determined in [3, 4]. In particular the cases (i) $a_n = const \in \mathbb{C}$, (ii) $a_n = -a_{n+1} \in \mathbb{C}$, (iii) $|a_{2n}| = 1,\ a_{2n+1} = i$, and (iv) $a_{2n} = a_{2m},\ a_{2n+1} = a_{2m+1}$ with $|a_n| = |a_m|\ \forall\, m, n \in \mathbb{N}$. In the present work we are interested in the convergence of $\mathcal{S}_1$ for arbitrary sequences of parameters. To evaluate $\mathcal{S}_1$, we study the recurrence for the classical convergents p_n/q_n to the fraction $\mathcal{S}_1$,

$$q_n = q_{n-1} + n^2\alpha_n q_{n-2} \quad \text{where} \quad \alpha_n := a_n^2. \tag{1.2}$$

We warn the reader that we will use α_n and a_n^2 interchangeably throughout. The p_n terms of the classical convergents also satisfy Eq.(1.2). Since the recurrence is a 2-step backward difference equation it is convenient to reformulate Eq.(1.2) in terms of 2×2 matrices

$$q^{(n)} = Q_n q^{(n-1)} \quad \text{where} \quad Q_n := \begin{bmatrix} 1 & n^2\alpha_n \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad q^{(n)} := \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}. \tag{1.3}$$

To analyze the case of cyclic parameters a_n with periods of length c, we regroup the above recursion into blocks of length c

$$q^{(cn+m)} = \widehat{Q}_n^{(m)} q^{(c(n-1)+m)} \quad \text{where} \quad \widehat{Q}_n^{(m)} := \prod_{j=c(n-1)+m+1}^{cn+m} Q_j. \tag{1.4}$$

Throughout, we interpret the matrix product ascending from right to left. To avoid notational clutter, we will, without loss of generality, consider only the 0th term of the cycle, i.e. $m = 0$ in Eq.(1.4). In this case $\widehat{Q}_n := \widehat{Q}_n^{(0)}$.

Following [4], it is helpful to consider the renormalized sequences (t_n) and (v_n) where

$$t_n := \frac{q_{n-1}}{n!} \quad \text{and} \quad v_n := \frac{q_n}{\Gamma(n+3/2)a_n^{(n+1)}}. \tag{1.5}$$

As with $q^{(n)}$ define

$$t^{(n)} := \begin{pmatrix} t_n \\ t_{n-1} \end{pmatrix} \quad \text{and} \quad v^{(n)} := \begin{pmatrix} v_n \\ v_{n-1} \end{pmatrix}. \tag{1.6}$$

Then

$$t^{(cn)} = T_n t^{(c(n-1))} \quad \text{where} \quad T_n := [N_n]^{-1}\widehat{Q}_n N_{n-1} \tag{1.7}$$

for

$$N_n := \mathrm{Diag}\left((cn)!, (cn-1)!\right). \tag{1.8}$$

Here we have also used the simplification of considering only the 0th term of the cycle in order to avoid cumbersome notation.

As we will see in Section 5 the sequence (v_n) lends itself to a more general analysis that is independent of the cycle length. Accordingly, we do not group the corresponding recurrence into cycles as above but study, instead, the base sequence

$$v^{(n)} = Y_n v^{(n-1)} \quad \text{where} \quad Y_n := G_n^{-1} Q_n G_{n-1} \tag{1.9}$$

for

$$G_n := \mathrm{Diag}\left(\Gamma\left(n+\frac{3}{2}\right) a_n^{(n+1)}, \Gamma\left(n+\frac{1}{2}\right) a_{n-1}^{n}\right). \tag{1.10}$$

By a standard identity [7, Eq.(1.2.10)], the separation of the convergents to $\mathcal{S}_1$ can be written as

$$\begin{aligned} \frac{p_{cn}}{q_{cn}} - \frac{p_{cn-1}}{q_{cn-1}} &= \frac{(-1)^{cn-1}(cn)!^2}{q_{cn}q_{cn-1}} \prod_{j=1}^{cn} a_j^2 \\ &= \frac{(-1)^{cn-1}(cn)!^2}{q_{cn}q_{cn-1}} \left(\prod_{j=1}^{c} \alpha_j\right)^n . \end{aligned} \tag{1.11}$$

In terms of the renormalized sequences (t_n), and (v_n), this is

$$\frac{p_{cn}}{q_{cn}} - \frac{p_{cn-1}}{q_{cn-1}} = \frac{(-1)^{cn-1}}{t_{cn+1}t_{cn}(cn+1)} \left(\prod_{j=1}^{c} \alpha_j\right)^n \tag{1.12}$$

$$= \frac{(-1)^{cn-1}}{v_{cn}v_{cn-1}a_{cn}^{(cn+1)} a_{cn-1}^{cn}} \left(\prod_{j=1}^{cn} \alpha_j\right) \left\{1 + O\left(\frac{1}{n}\right)\right\}. \tag{1.13}$$

Thus, for $|a_n| = |a_m| = b \neq 0$ for all $n, m \in \mathbb{N}$, the continued fraction $\mathcal{S}_1$ diverges – that is, the convergents separate – if

$$|t_n| \leq O\left(\frac{b^n}{\sqrt{n}}\right) \quad \text{or} \quad (v_n) \text{ is bounded,} \tag{1.14}$$

each of these being equivalent.

To begin, we focus our attention on cyclic parameters, that is $a_{n+c} = a_n$ for $c \geq 1$ and all n. We then broaden the analysis to infinite and random sequences. In Section 4 we investigate the first of the convergence criterion Eq.(1.14) using operator norms. The advantage of this criterion is that it yields an asymptotic estimate of the rate of divergence of $\mathcal{S}_1$ for the cases we consider. The second criterion allows for more general, albeit less detailed, analytical techniques, which we apply in Section 5. The case of random sequences (a_n) builds upon the ideas developed in Section 5 but requires a slightly different perspective which we develop in Section 6. A summary of our most attractive results is given in Theorem 7.1. Before proceeding with the analysis, however, we motivate this study in Section 2 with some numerical experiments of specific examples.