CATEGORY THEORY

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1 · Categories, functors and natural transformations

1.1 · *Categories*

DEFINITION 1.1.1

A *category* $\mathcal{C}$ consists of:

- a collection of objects, ob $\mathcal{C}$;
- For every pair $X, Y \in \text{ob}\,\mathcal{C}$, a collection $\mathcal{C}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$ of morphisms $f\colon X \longrightarrow Y$,

equipped with:

- for each $X \in \text{ob}\,\mathcal{C}$, an identity map $\text{id}_X = 1_X \in \mathcal{C}(X, X)$;
- for each $X, Y, Z \in \text{ob}\,\mathcal{C}$, a composition map

$$\begin{aligned} m_{XYZ}\colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) &\longrightarrow \mathcal{C}(X, Z) \\ (g, f) &\longmapsto g \circ f = gf, \end{aligned}$$

satisfying:

- unit laws — if $f\colon X \longrightarrow Y$ then $1_Y \circ f = f = f \circ 1_X$
- associativity — if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, then $h(gf) = (hg)f$.

A category is said to be *small* if ob $\mathcal{C}$ and all of the $\mathcal{C}(X, Y)$ are sets, and *locally small* if each $\mathcal{C}(X, Y)$ is a set.

REMARKS

1. If $f \in \mathcal{C}(X, Y)$, we say that X and Y are the *domain* (or *source*) and the *codomain* (or *target*) of f.
2. Morphisms are also referred to as *maps* or *arrows*.
3. We can write $\text{Hom}_{\mathcal{C}}$ for the collection of all morphisms.
4. It is convenient and customary to assume that the $\mathcal{C}(X, Y)$ are disjoint for distinct pairs (X, Y).
5. We don't worry ourselves with the niceties of set theory.

DEFINITION 1.1.2

A category $\mathcal{C}$ is called *discrete* if the only morphisms are identities; i.e.

$$\mathcal{C}(X, Y) = \begin{cases} \{1_X\} & \text{if } X = Y \\ \varnothing & \text{otherwise.} \end{cases}$$

EXAMPLES 1.1.3

1. Large categories of mathematical structures:
   a. **Set** of sets and functions.
   b. Categories derived from or related to **Set**:

- **Pfn** of sets and partial functions;
- **Rel** of sets and relations;
- **Set**$_*$ of pointed sets and base point preserving functions.

c Algebraic structures and structure-preserving maps:

- **Grp** of groups and group homomorphisms;
- **Ab** of abelian groups and group homomorphisms;
- **Ring** of rings and ring homomorphisms;
- **Vec** of vector spaces over $\mathbb{R}$;
- **Mat** of natural numbers and $n \times m$ matrices.

d Topological categories:

- **Top** of topological spaces and continuous maps;
- **Haus** of Hausdorff spaces and continuous maps;
- **Met** of metric spaces and uniformly continuous maps;
- **Htpy** of topological spaces and homotopy classes of maps.

2 Mathematical structures as categories:

a Posets: a poset $(P, \leqslant)$ can be regarded as a category $\mathcal{C}$ with objects the elements of P and precisely one morphism $x \longrightarrow y$ when $x \leqslant y$ and none otherwise.
b Monoids: a category with just one object is a monoid.
c Groups: a group G can be regarded as a category with just one (formal) object and whose morphisms are the elements of G.

3 Small categories can be presented by generators and relations. From a directed graph we can generate a category of "paths through the graph" and then add relations imposing equalities between some paths with the same domain and codomain.

a There is a category **0** with no objects and no morphisms, generated by the empty graph.
b There is a category **1** with one objects and one (identity) morphism, generated by the graph with just one vertex.
c There is a category generated by the graph with one vertex and one edge. It is isomorphic to the additive monoid $\mathbb{N}$.
d There is a category generated by the graph with one vertex and one edge s say, together with the relation $s^2 = 1$. It has one object and two morphisms and is isomorphic to the cyclic group of order 2.
e There is a category generated by the graph with two vertices and one edge between them. It has two objects and three morphisms and is isomorphic to the poset $\mathbf{2} = \{0 \leqslant 1\}$.

1.2 · Universal properties

DEFINITION 1.2.1

A morphism $f \in \mathcal{C}(X, Y)$ is an *isomorphism* if $\exists g \in \mathcal{C}(Y, X)$ such that $gf = 1_X$ and $fg = 1_Y$. We say g is an *inverse* for f.

PROPOSITION 1.2.2

If g_1 and g_2 are inverses for f, then $g_1 = g_2$.

PROOF

$g_1 = g_1 \circ 1_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = 1_X \circ g_2 = g_2$. □

PROPOSITION 1.2.3

1 The identity map is an isomorphism.
2 The composition of two isomorphisms is an isomorphism.

PROOF

1 1_X is clearly self-inverse.
2 Let $f \in \mathcal{C}(Y, Z)$, $g \in \mathcal{C}(X, Y)$ be isomorphisms, with respective inverses $h \in \mathcal{C}(Z, Y)$, $k \in \mathcal{C}(Y, X)$. Then we claim that $fg \in \mathcal{C}(X, Z)$ is an isomorphism, with inverse $kh \in \mathcal{C}(Z, X)$. For

$$(fg)(kh) = f(gk)h = f(1_Y)h = fh = 1_Z$$
$$(kh)(fg) = k(hf)g = k(1_Y)g = kg = 1_X$$

so we have the desired result. □

DEFINITION 1.2.4

A *terminal object* in $\mathcal{C}$ is an element $T \in \text{ob}\,\mathcal{C}$ such that $\forall X \in \mathcal{C}$, $\exists!$ morphism $X \xrightarrow{k} T$.

EXAMPLE

In **Set**, every 1-element set is terminal. So sometimes we denote a terminal object by 1.

PROPOSITION 1.2.5

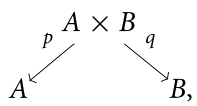
Suppose 1 and $1'$ are terminal in $\mathcal{C}$. Then there exists a unique isomorphism $f \in \mathcal{C}(1, 1')$.

PROOF

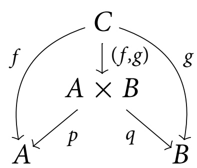
Since $1'$ is terminal, there is a unique morphism $f\colon 1 \longrightarrow 1'$. Similarly, 1 is terminal, so there is a unique morphism $f'\colon 1' \longrightarrow 1$. Now consider $f' \circ f \in \mathcal{C}(1, 1)$. Since 1 is terminal, there is a unique morphism $1 \longrightarrow 1$, i.e. the identity. So $f' \circ f = \text{id}_1$; similarly $f \circ f' = \text{id}_{1'}$. Hence f is the desired unique isomorphism. □

DEFINITION 1.2.6

Given $A, B \in \text{ob}\,\mathcal{C}$, a *product* of A and B is an object $A \times B$ equipped with projections



such that for all $f\colon C \longrightarrow A$, $g\colon C \longrightarrow B$, $\exists!$ morphism $(f, g)\colon C \longrightarrow A \times B$ such that $p \circ (f, g) = f$ and $q \circ (f, g) = g$; i.e. such that



commutes.

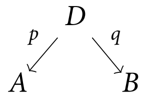
EXAMPLE

In **Set**, $A \times B = \{\, (a, b) \mid a \in A, b \in B \,\}$ with p, q the first and second projections.

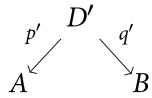
Note however, that we could also have taken p, q to be the second and first projections, or the set to be $\{ (b, a) \mid b \in B, a \in A \}$.

PROPOSITION 1.2.7

If



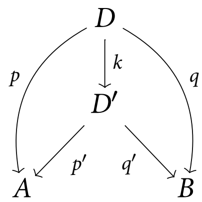
and

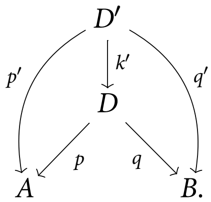


are products of $A, B \in \mathcal{C}$, then $\exists!$ isomorphism $k\colon D \longrightarrow D'$ such that $q'k = q$ and $p'k = p$.

PROOF

Consider the diagrams



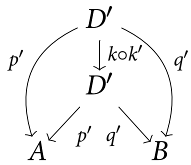


By our definition of product, k is the unique morphism $D \longrightarrow D'$ s.t. these diagrams commute; so $q'k = q$ and $p'k = p$ certainly.

We claim that k' is an inverse for k. For consider $k \circ k'\colon D' \longrightarrow D'$. We have

$$p' \circ (k \circ k') = (p' \circ k) \circ k' = p \circ k' = p'$$
$$q' \circ (k \circ k') = (q' \circ k) \circ k' = q \circ k' = q'$$

Hence



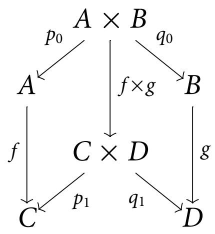
commutes. But by the definition of product, there is a unique morphism $D' \longrightarrow D'$ that makes this diagram commute, i.e. the identity. So $k \circ k' = \mathrm{id}_{D'}$. Similarly $k' \circ k = \mathrm{id}_D$. So k is indeed an isomorphism, and is the unique one s.t. $q'k = q$ and $p'k = p$. □

DEFINITION 1.2.8

If $\forall A, B \in \mathcal{C}$, there exists a product $A \times B$, we say $\mathcal{C}$ has all binary products.

PROPOSITION 1.2.9

If $\mathcal{C}$ is a category with binary products, then given $f \in \mathcal{C}(A, C)$, $g \in \mathcal{C}(B, D)$, there exists a unique morphism $f \times g \in \mathcal{C}(A \times B, C \times D)$ such that



commutes.

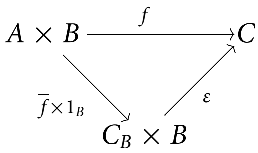
PROOF

Immediate from definition of product. □

LECTURE 3 · 16/10/02

DEFINITION 1.2.10

Suppose $\mathcal{C}$ is a category with binary products. Given $B, C \in \operatorname{ob}\mathcal{C}$, a *function space* or *exponential* is an object C^B equipped with an *evaluation morphism* $\varepsilon\colon C^B \times B \longrightarrow C$ such that $\forall f\colon A \times B \longrightarrow C$, $\exists!\ \overline{f}\colon A \longrightarrow C^B$ such that



commutes, i.e. $\varepsilon \circ (\overline{f} \times 1_B) = f$.

In **Set**, $C^B = \{ f\colon B \longrightarrow C \} = [B, C]$. There is an evaluation map

$$\begin{aligned} \varepsilon\colon C^B \times B &\longrightarrow C \\ (g, b) &\longmapsto g(b). \end{aligned}$$

Given $f\colon A \times B \longrightarrow C$, fix $a \in A$ to get

$$\begin{aligned} f_a\colon B &\longrightarrow C \\ b &\longmapsto f(a, b). \end{aligned}$$

So we have a function

$$\begin{aligned} \overline{f}\colon A &\longrightarrow C^B \\ a &\longmapsto f_a, \end{aligned}$$

such that

$$\begin{aligned} f(a, b) &= f_a(b) \\ &= \varepsilon(f_a, b) \\ &= \varepsilon \circ (\overline{f} \times 1_B)(a, b). \end{aligned}$$

So $\varepsilon \circ (\overline{f} \times 1_B) = f$ as required.

1.3 · *Categorical constructions*

DEFINITION 1.3.1

A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of subcollections

- $\mathrm{ob}\,\mathcal{D} \subseteq \mathrm{ob}\,\mathcal{C}$;
- $\mathrm{Hom}_{\mathcal{D}} \subseteq \mathrm{Hom}_{\mathcal{C}}$,

together with composition and identities inherited from $\mathcal{C}$. We say $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$ if $\forall X, Y \in \mathcal{D}$, $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)$, and a *lluf subcategory* of $\mathcal{C}$ if $\mathrm{ob}\,\mathcal{C} = \mathrm{ob}\,\mathcal{D}$.

We can think of the data for a category as

$$\mathrm{Hom}_{\mathcal{C}} \underset{c_2}{\overset{c_1}{\rightrightarrows}} \mathrm{ob}\,\mathcal{C}$$

We could have c_1 giving us the domain of a morphism and c_2 the codomain, or vice verse. This motivates the definition:

DEFINITION 1.3.2

Given a category $\mathcal{C}$, the *dual* or *opposite* category $\mathcal{C}^{\mathrm{op}}$ is defined by:–

- $\mathrm{ob}\,\mathcal{C} = \mathrm{ob}\,\mathcal{C}^{\mathrm{op}}$;
- $\mathcal{C}(X, Y) = \mathcal{C}^{\mathrm{op}}(Y, X)$;
- identities inherited;
- $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$.

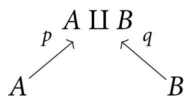
THE PRINCIPLE OF DUALITY

Given any property, feature or theorem in terms of diagrams of morphisms, we can immediately obtain its dual by reversing all the arrows (this is often indicated by the prefix "co-").

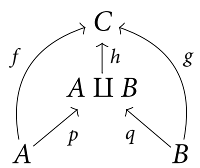
EXAMPLES 1.3.3

1 The dual notion of a terminal category object is an *initial* object. That is, an object $I \in \mathcal{C}$ such that for all $Y \in \mathcal{C}$, there exists a unique $f\colon I \longrightarrow Y$. For example, the (unique) initial object in **Set** is $\varnothing$; we sometimes write 0 for an initial object.

2 The dual of a product is a *coproduct*:



where p, q are *coprojections* such that, for any $f \in \mathcal{C}(A, C)$, $g \in \mathcal{C}(B, C)$, $\exists!\, h\colon A \amalg B \longrightarrow C$ such that



commutes.

DEFINITION 1.3.4

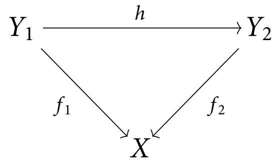
A morphism $A \overset{m}{\longrightarrow} B$ is *monic* iff given any $f, g\colon C \longrightarrow A$, we have $mf = mg \Rightarrow f = g$. Dually, a morphism $A \overset{e}{\longrightarrow} B$ is *epic* iff given any $f, g\colon B \longrightarrow C$, we have $fe = ge \Rightarrow f = g$.

It is easy to see that any isomorphism is epic and monic. In **Set**, a morphism is monic iff it is injective, and epic iff it is surjective.

DEFINITION 1.3.5

Given $\mathcal{C}$ a category and $X \in \text{ob}\,\mathcal{C}$, then the *slice over* X, $\mathcal{C}/X$ is the category with:

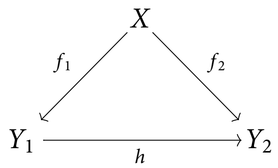
- objects (Y, f), where $f\colon Y \longrightarrow X \in \mathcal{C}$;
- morphisms $h\colon (Y_1, f_1) \longrightarrow (Y_2, f_2)$ such that



commutes, i.e. $f_2 h = f_1$.

Dually, we have the *slice under* X, $X/\mathcal{C}$, with:

- objects (Y, f), where $f\colon X \longrightarrow Y \in \mathcal{C}$;
- morphisms $h\colon (Y_1, f_1) \longrightarrow (Y_2, f_2)$ such that



commutes, i.e. $hf_1 = f_2$.

We have a terminal object $(X, 1_X)$ in $\mathcal{C}/X$ and dually an initial object $(X, 1_X)$ in $X/\mathcal{C}$.

1.4 · *Functors*

DEFINITION 1.4.1

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ associates

- with each $X \in \text{ob}\,\mathcal{C}$, an object $FX \in \text{ob}\,\mathcal{D}$;
- with each $f \in \mathcal{C}(X, Y)$, a morphism $Ff \in \mathcal{D}(FX, FY)$,

such that

- $F1_X = 1_{FX}$;
- $F(gf) = Fg \circ Ff$.

LECTURE 4 · 18/10/02

DEFINITION 1.4.2

We define the category **Cat** of small categories:-

- For any category $\mathcal{C}$ there is an identity functor

$$\begin{aligned} 1_{\mathcal{C}}\colon \mathcal{C} &\longrightarrow \mathcal{C} \\ X &\longmapsto X \\ f &\longmapsto f \end{aligned}$$

- Composition of functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ with GF defined in the obvious way.

  Similarly we have **CAT**, the category of large categories and functors.

EXAMPLES 1.4.3

1 **Cat** has an initial object 0.
2 **Cat** has a terminal object 1.
3 **Cat** has products; given $\mathcal{C}, \mathcal{D} \in \text{ob}\,\mathbf{Cat}$, we have the product $\mathcal{C} \times \mathcal{D}$ with

   - objects (c, d), $c \in \mathcal{C}$, $d \in \mathcal{D}$;
   - morphisms (f, g), $f\colon c \longrightarrow c' \in \mathcal{C}$, $g\colon d \longrightarrow d' \in \mathcal{D}$.

DEFINITION 1.4.4

A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is *faithful/full/full and faithful* if $\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$ is injective/surjective/an isomorphism.

EXAMPLES 1.4.5

1 Functors between collections of mathematical objects:

   a forgetful functors:

$$\begin{aligned} \mathbf{Gp} &\to \mathbf{Set} \\ \mathbf{Ring} &\to \mathbf{Set} \\ \mathbf{Ring} &\to \mathbf{Ab} \\ \mathbf{Haus} &\to \mathbf{Top}; \end{aligned}$$

   b free functors:

$$\begin{aligned} \mathbf{Set} &\to \mathbf{Gp} \\ \mathbf{Set} &\to \mathbf{Mnd}; \end{aligned}$$

   c inclusion of subcategories:

$$\begin{aligned} \mathbf{Ab} &\to \mathbf{Gp} \\ \mathbf{Haus} &\to \mathbf{Top}. \end{aligned}$$

2 Functors between mathematical structures:

   a posets $f\colon (P, \leqslant) \longrightarrow (Q, \preccurlyeq)$ is an order-preserving map;
   b groups $f\colon G \longrightarrow H$ is a group homomorphism.

3 Presheaves – a functor $\mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set}$ is called a *presheaf* on $\mathcal{C}$.
4 Diagrams – a functor $\mathcal{C} \longrightarrow \mathbf{Set}$ is called a *diagram* on $\mathcal{C}$.

Note that a functor will preserve any property that is expressible as a commutative diagram. For example, isomorphisms are preserved by all functors; if f is an isomorphism, then Ff is also.

PROPOSITION

If F is full and faithful, then Ff isomorphic $\Leftrightarrow$ f isomorphic.

PROOF

Let $f \in \mathcal{C}(X, Y)$ such that Ff is an isomorphism. Then $\exists$ inverse $g' \in \mathcal{D}(FY, FX)$ for Ff. Since F is full, then $\exists g \in \mathcal{C}(Y, X)$ such that $g' = Fg$. But now

$$F(fg) = (Ff)(Fg) = 1_{FY}.$$

And $F(1_Y) = 1_{FY}$, so since F is faithful, we have $fg = 1_Y$. Similarly $gf = 1_X$. So g is an inverse for $f \in \mathcal{C}(X, Y)$, i.e. f is an isomorphism. □

1.5 · *Contravariant functors*

DEFINITION 1.5.1

A *contravariant* functor $\mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{\text{op}} \to \mathcal{D}$. That is:

- on objects, $X \mapsto FX$;
- on morphisms, $X \xrightarrow{f} Y \mapsto FY \xrightarrow{Ff} FX$;
- identities are preserved;
- $F(g \circ f) = Ff \circ Fg$.

A non-contravariant functor is sometimes referred to as a *covariant* functor.

1.6 · *The Hom functor*

1.6.1 · REPRESENTABLES

Let $\mathcal{C}$ be a locally small category. We have a contravariant functor H_U or $\mathcal{C}(\_, U)$:

$$\begin{aligned} H_U \colon \mathcal{C}^{\text{op}} &\longrightarrow \textbf{Set} \\ X &\longmapsto \mathcal{C}(X, U) \end{aligned}$$

$$\begin{array}{ccccc} X & & \mathcal{C}(X, U) & & g \\ {\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle \mathcal{C}(1,g)} & & \downarrow \\ Y & & \mathcal{C}(Y, U) & & gf \end{array}$$

Dually, we have a covariant functor H^U or $\mathcal{C}(U, \_)$:

$$\begin{aligned} H^U \colon \mathcal{C} &\longrightarrow \textbf{Set} \\ X &\longmapsto \mathcal{C}(U, X) \end{aligned}$$

$$\begin{array}{ccccc} X & & \mathcal{C}(U, X) & & g \\ {\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle \mathcal{C}(f,1)} & & \downarrow \\ Y & & \mathcal{C}(U, Y) & & fg \end{array}$$

These are known as *representables*.

1.6.2 · THE Hom FUNCTOR

Again, take $\mathcal{C}$ locally small. Then we have a functor

$$\begin{aligned} H \colon \mathcal{C}^{\text{op}} \times \mathcal{C} &\longrightarrow \textbf{Set} \\ (X, Y) &\longmapsto \mathcal{C}(X, Y) \end{aligned}$$

$$\begin{array}{ccccc} (X, Y) & & \mathcal{C}(X, Y) & & h \\ {\scriptstyle (f,g)}\downarrow & \longmapsto & \downarrow{\scriptstyle \mathcal{C}(f,g)} & & \downarrow \\ (X', Y') & & \mathcal{C}(X', Y') & & ghf \end{array}$$

where $f \colon X \to X' \in \mathcal{C}^{\text{op}}$ and $g \colon Y \to Y' \in \mathcal{C}$.

1.7 · *Natural transformations*

DEFINITION 1.7.1

Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A *natural transformation* $\alpha\colon F \longrightarrow G$ is a collection of morphisms (known as *components*)

$$\{\, \alpha_X\colon FX \longrightarrow GX \mid X \in \mathcal{C} \,\},$$

such that, $\forall f\colon X \longrightarrow Y \in \mathcal{C}$,

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array}$$

commutes (the *naturality condition*).

LECTURE 5 · 21/10/02

DEFINITION 1.7.2

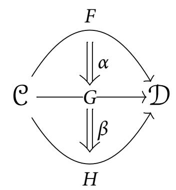
Given categories $\mathcal{C}$ and $\mathcal{D}$, we define the (larger) category $[\mathcal{C}, \mathcal{D}]$ where:

- objects are functors $F\colon \mathcal{C} \longrightarrow \mathcal{D}$;
- morphisms are natural transformations $\alpha\colon F \longrightarrow G$,

such that:

- identities are natural transformations $1_F\colon F \longrightarrow F$ (for any $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ with components $FX \xrightarrow{1_{FX}} FX$;
- for composition, given $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$, then $\beta \circ \alpha$ is the natural transformation with components

$$(\beta \circ \alpha)_X\colon FX \xrightarrow{\beta_X \circ \alpha_X} HX.$$



So, for example, $[\mathcal{C}, \mathcal{D}](F, G)$ is a collection of natural transformations $F \longrightarrow G$.

DEFINITION 1.7.3

A *natural isomorphism* $\alpha\colon F \longrightarrow G$ is an isomorphism in the functor category; i.e. there exists $\beta\colon G \longrightarrow F$ such that $\alpha \circ \beta = 1_G$ and $\beta \circ \alpha = 1_F$. Note that two natural transformations are equal iff all their components are.

PROPOSITION 1.7.4

$\alpha\colon F \longrightarrow G$ is a natural isomorphism iff each component $\alpha_X\colon FX \longrightarrow GX$ is an isomorphism in $\mathcal{D}$.

PROOF

Suppose α is a natural isomorphism, and let β be its inverse. Then

$$\alpha \circ \beta = 1_G \quad \Rightarrow \quad (\alpha \circ \beta)_X = 1_{GX} \quad \Rightarrow \quad \alpha_X \circ \beta_X = 1_{GX}$$

and

$$\beta \circ \alpha = 1_F \quad \Rightarrow \quad (\beta \circ \alpha)_X = 1_{FX} \quad \Rightarrow \quad \beta_X \circ \alpha_X = 1_{FX}.$$

So β_X is an inverse for α_X for each $X \in \mathcal{C}$. Thus each component is an isomorphism in $\mathcal{D}$.

Conversely, if each component α_X is an isomorphism, then let β_X be the corresponding inverses for each $X \in \mathcal{C}$. Now, given $f \in \mathcal{C}(X, Y)$, we have that

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array}$$

commutes; i.e. $(Gf) \circ \alpha_X = \alpha_Y \circ (Ff)$. But now:–

$$\begin{aligned} \beta_Y \circ (Gf) \circ \alpha_X \circ \beta_X &= \beta_Y \circ \alpha_Y \circ (Ff) \circ \beta_X \\ \text{so} \quad \beta_Y \circ (Gf) \circ 1_{GX} &= 1_{FY} \circ (Ff) \circ \beta_X \\ \text{so} \quad \beta_Y \circ (Gf) &= (Ff) \circ \beta_X; \end{aligned}$$

hence

$$\begin{array}{ccc} GX & \xrightarrow{\beta_X} & FX \\ {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\ GY & \xrightarrow[\beta_Y]{} & FY \end{array}$$

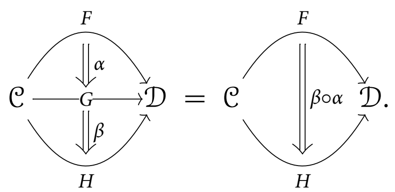
commutes; so we can legitimately define the natural transformation β with components β_X. And clearly β is an inverse for α, so α is a natural isomorphism. □

We can prove similar results that tell us that α is epic/monic iff all its components are.

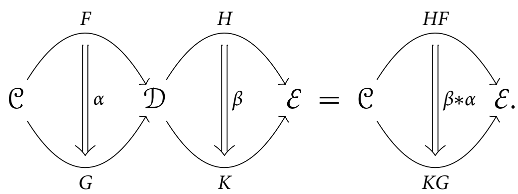
1.8 · *The 2-category **Cat***

DEFINITION 1.8.1

We define *"horizontal composition"* of natural transformations. We have seen "vertical composition" already:



But we can also compose:



We define $(\beta * \alpha)_X \colon HFX \longrightarrow KGX$ by

$$HFX \xrightarrow{H\alpha_X} HGX \xrightarrow{\beta_{GX}} KGX$$

or

$$HFX \xrightarrow{\beta_{FX}} KFX \xrightarrow{K\alpha_X} KGX.$$

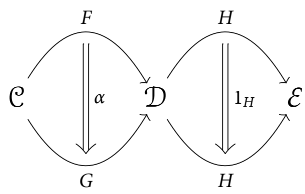
By the naturality of β, these definitions are equivalent:

$$\begin{array}{ccc} HFX & \xrightarrow{\beta_{FX}} & KFX \\ {\scriptstyle H\alpha_X}\downarrow & & \downarrow{\scriptstyle K\alpha_X} \\ HGX & \xrightarrow[\beta_{GX}]{} & KGX \end{array}$$

so we can define

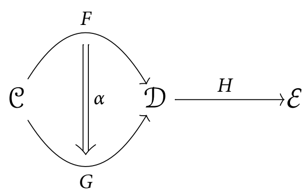
$$(\beta * \alpha)_X = \beta_{GX} \circ H\alpha_X = K\alpha_X \circ \beta_{FX}.$$

We consider the following particular case:



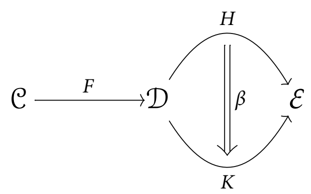
$$1_H * \alpha \colon HF \to HG$$

which we will (for convenience) write as:



$$H\alpha \colon HF \to HG.$$

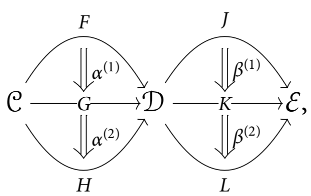
Similarly we have:



$$\beta F \colon HF \to KF.$$

PROPOSITION 1.8.2 (THE MIDDLE-4 INTERCHANGE LAW)

Given



we have $(\beta^{(2)} \circ \beta^{(1)}) * (\alpha^{(2)} \circ \alpha^{(1)}) = (\beta^{(2)} * \alpha^{(2)}) \circ (\beta^{(1)} * \alpha^{(1)})$.

PROOF

Consider components. We have

$$\begin{aligned}[(\beta^{(2)} \circ \beta^{(1)}) * (\alpha^{(2)} \circ \alpha^{(1)})]_X &= (\beta^{(2)} \circ \beta^{(1)})_{HX} \circ J(\alpha^{(2)} \circ \alpha^{(1)})_X \\ &= \beta^{(2)}_{HX} \circ \beta^{(1)}_{HX} \circ J\alpha^{(2)}_X \circ J\alpha^{(1)}_X\end{aligned}$$

and

$$[(\beta^{(2)} * \alpha^{(2)}) \circ (\beta^{(1)} * \alpha^{(1)})]_X = \beta^{(2)}_{HX} \circ K\alpha^{(2)}_X \circ \beta^{(1)}_{GX} \circ J\alpha^{(1)}_X.$$

So it is sufficient to prove that $K\alpha^{(2)}_X \circ \beta^{(1)}_{GX} = \beta^{(1)}_{HX} \circ J\alpha^{(2)}_X$. But we have that

$$\begin{array}{ccc} JGX & \xrightarrow{\beta^{(1)}_{GX}} & KGX \\ {\scriptstyle J\alpha^{(2)}_X}\downarrow & & \downarrow{\scriptstyle K\alpha^{(2)}_X} \\ JHX & \xrightarrow[\beta^{(1)}_{HX}]{} & KHX \end{array}$$

commutes (by the naturality of $\beta^{(1)}$), and so we are done. □

DEFINITION 1.8.3

We can now define the 2-category **Cat**, consisting of:

- objects, morphisms and two-cells;
- composition of morphisms;
- horizontal and vertical composition of 2-cells;
- axioms - unit, associativity and middle-4 interchange; “any two ways of composing are the same”.

DEFINITION 1.8.4

Given categories $\mathcal{C}$ and $\mathcal{D}$, an *equivalence* consists of:

- functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$;
- natural isomorphisms $GF \overset{\alpha}{\Rightarrow} 1_{\mathcal{C}}$, $FG \overset{\beta}{\Rightarrow} 1_{\mathcal{D}}$.

We call β the *inverse up to isomorphism* or the *pseudo-inverse* of α.

DEFINITION 1.8.5

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* on objects iff $\forall\ Y \in \mathcal{D}$, $\exists\ X \in \mathcal{C}$ such that $FX \cong Y \in \mathcal{D}$.

PROPOSITION 1.8.6

F is an equivalence of categories iff it is essentially surjective and full and faithful.

PROOF

Omitted. □

2 · Representability

2.1 · *The Yoneda Embedding*

Recall that for each $A \in \mathcal{C}$, we have the functor $H_A\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. So we have an assignation $A \mapsto H_A$. We can extend this to a functor, known as the *Yoneda embedding*:–

$$\begin{aligned} H_\bullet\colon \mathcal{C} &\longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \\ A &\longmapsto H_A \\ (f\colon A \longrightarrow B) &\longmapsto (H_f\colon H_A \longrightarrow H_B), \end{aligned}$$

where H_f is the natural transformation with components

$$\begin{aligned} (H_f)_X\colon H_A X &\longrightarrow H_B X \\ \text{i.e. } \mathcal{C}(X, A) &\longrightarrow \mathcal{C}(X, B) \\ h &\longmapsto f \circ h. \end{aligned}$$

We need to check that this is a well-defined natural transformation, i.e. that

$$\begin{array}{ccc} \mathcal{C}(Y, A) & \xrightarrow{(H_f)_Y = f\circ\_} & \mathcal{C}(Y, B) \\ {\scriptstyle H_A g = \_\circ g}\downarrow & & \downarrow{\scriptstyle H_B g = \_\circ g} \\ \mathcal{C}(X, A) & \xrightarrow[(H_f)_X = f\circ\_]{} & \mathcal{C}(X, B) \end{array}$$

commutes. But along the two legs we just have:–

$$\begin{array}{ccc} h & \longmapsto & f \circ h \\ & & \downarrow \\ & & (f \circ h) \circ g \end{array} \qquad \text{and} \qquad \begin{array}{ccc} h & & \\ \downarrow & & \\ h \circ g & \longmapsto & f \circ (h \circ g) \end{array}$$

so the naturality condition just says that composition is associative.

LECTURE 6 · 23/10/02

2.2 · *Representable Functors*

DEFINITION 2.2.1

A functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is *representable* if it is naturally isomorphic to H_A for some $A \in \mathcal{C}$, and a *representation* for F is an object $A \in \mathcal{C}$ together with a natural isomorphism $\alpha\colon H_A \longrightarrow F$.

Dually, a functor $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ is representable if $F \cong H^A$ for some $A \in \mathcal{C}$, and a representation for F is an object A with a natural isomorphism $\alpha\colon H^A \longrightarrow F$.

NOTE

The naturality square says, that $\forall f\colon V \longrightarrow W \in \mathcal{C}$,

$$\begin{array}{ccc} \mathcal{C}(W, A) & \xrightarrow{\alpha_W} & FW \\ {\scriptstyle H_A f = \_ \circ f}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathcal{C}(V, A) & \xrightarrow[\alpha_V]{} & FV \end{array}$$

commutes.

EXAMPLES 2.2.2

1 The forgetful functor $U\colon \mathbf{Gp} \longrightarrow \mathbf{Set}$ is representable. Take $A = \mathbb{Z}$, and α to be the natural transformation with components:

$$\begin{aligned} \alpha_G\colon H^{\mathbb{Z}} G &\longrightarrow UG \\ f &\longmapsto f(1). \end{aligned}$$

Then we can check that α is natural, and it is an isomorphism, since any homomorphism $f\colon \mathbb{Z} \longrightarrow G$ is completely determined by $f(1)$.

2 ob: $\mathbf{Cat} \longrightarrow \mathbf{Set}$ is representable. For let A be 1, the terminal category; then $\mathrm{ob}(\mathcal{C}) \cong \mathbf{Cat}(1, \mathcal{C})$ is a natural isomorphism.

Now, we can make a few suggestive observations about natural transformations $\alpha\colon H_A \longrightarrow F$. Consider the naturality square

$$\begin{array}{ccc} \mathcal{C}(A, A) & \xrightarrow{\alpha_A} & FA \\ {\scriptstyle \_ \circ f}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathcal{C}(V, A) & \xrightarrow[\alpha_V]{} & FV \end{array}$$

We know this commutes; in particular, for the element $1_A \in \mathcal{C}(A, A)$, we have

$$\alpha_V(1_A \circ f) = Ff(\alpha_A(1_A)),$$

so that α is in fact completely determined by $\alpha_A(1_A) \in FA$. So, we would like to define a natural transformation $\alpha\colon H_A \longrightarrow F$ by setting $\alpha(1_A) = x \in FA$, and $\alpha_V(f) = (Ff)(x)$. If this is indeed a natural transformation, then we will have set up a bijection between FA and the natural transformations $H_A \longrightarrow F$. Hence we get …

2.3 · *The Yoneda Lemma*

THEOREM 2.3.1 (YONEDA LEMMA)

Let $\mathcal{C}$ be a locally small category, $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. Then there is an isomorphism

$$FA \cong [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F),$$

which is natural in A and F; i.e.

$$\begin{array}{ccc} FB & \longrightarrow & [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_B, F) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle \_\circ H_f} \\ FA & \longrightarrow & [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F) \end{array} \quad \text{and} \quad \begin{array}{ccc} FA & \longrightarrow & [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F) \\ {\scriptstyle \theta_A}\downarrow & & \downarrow{\scriptstyle \theta\circ\_} \\ GA & \longrightarrow & [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, G) \end{array}$$

commute, for all $f\colon A \longrightarrow B$ and for all $\theta\colon F \longrightarrow G$ respectively.

PROOF

1 Given $x \in FA$, we define $\widehat{x} \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)$ by components:

$$\begin{aligned} \widehat{X}_V\colon \mathcal{C}(V, A) &\longrightarrow FV \\ f &\longmapsto Ff(x) \end{aligned}$$

We must check the naturality of $\widehat{x}$; given $g\colon W \longrightarrow V$, we need

$$\begin{array}{ccc} \mathcal{C}(V, A) & \xrightarrow{\widehat{x}_V} & FV \\ {\scriptstyle \_\circ g}\downarrow & & \downarrow{\scriptstyle Fg} \\ \mathcal{C}(W, A) & \xrightarrow[\widehat{x}_W]{} & FW \end{array}$$

to commute. On elements, we have

$$\begin{array}{ccc} f & \longmapsto & Ff(x) \\ & & \downarrow \\ & & Fg\left(Ff(x)\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} f & & \\ \downarrow & & \\ f\circ g & \longmapsto & F(f\circ g)(x) \end{array}$$

But $Fg(Ff(x)) = F(f \circ g)(x)$ by the (contravariant) functoriality of F, so the square commutes as required.

2 Given $\alpha \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)$, we define $\widehat{\alpha} \in FA$ by

$$\widehat{\alpha} = \alpha_A(1_A).$$

3 We check $(\widehat{\widehat{\ }}) = (\ )$. Given $x \in FA$,

$$\begin{aligned} \widehat{\widehat{x}} = \widehat{x}_A(1_A) &= F(1_A)(x) \\ &= 1_{FA}(x) \\ &= x. \end{aligned}$$

Given $\alpha \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)$, $\widehat{\widehat{\alpha}}$ is given by components

$$\begin{aligned} \widehat{\widehat{\alpha}}\colon \mathcal{C}(V, A) &\longrightarrow FV \\ f &\longmapsto Ff(\widehat{\alpha}) = Ff(\alpha_A(1_A)). \end{aligned}$$

So we need only check that $\alpha_V(f) = Ff(\alpha_A(1_A))$. We have the following naturality square for α:

$$\begin{array}{ccc} \mathcal{C}(A,A) & \xrightarrow{\alpha_A} & FA \\ {\scriptstyle \_\circ f}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathcal{C}(V,A) & \xrightarrow[\alpha_V]{} & FV \end{array}$$

so on the element $1_A \in \mathcal{C}(A,A)$, we have $\alpha_V(1_A \circ f) = Ff(\alpha_A(1_A))$, as required.

4 We check naturality in A, i.e. that given any $B \xrightarrow{f} A$,

$$\begin{array}{ccc} FA & \xrightarrow{\widehat{\ }} & [\mathcal{C}^{\text{op}}, \mathbf{Set}](H_A, F) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle \_\circ H_f} \\ FB & \xrightarrow{\widehat{\ }} & [\mathcal{C}^{\text{op}}, \mathbf{Set}](H_B, F) \end{array}$$

commutes. On elements, we have:

$$\begin{array}{ccc} x & \longmapsto & \widehat{x} \\ & & \downarrow \\ & & \widehat{x} \circ H_f \end{array} \qquad \text{and} \qquad \begin{array}{ccc} x & & \\ \downarrow & & \\ Ff(x) & \longmapsto & \widehat{Ff(x)}. \end{array}$$

Now, the former has components

$$\begin{array}{ccccc} \mathcal{C}(V,B) & \xrightarrow{(H_f)_V} & \mathcal{C}(V,A) & \xrightarrow{\widehat{x}_V} & FV \\ g & \longmapsto & f \circ g & \longmapsto & F(f \circ g)(x), \end{array}$$

and the latter

$$\begin{array}{ccc} \mathcal{C}(V,B) & \xrightarrow{\widehat{Ff(x)}_V} & FV \\ g & \longmapsto & Fg \circ Ff(x). \end{array}$$

But $(Fg \circ Ff)(x) = F(f \circ g)(x)$ by the functoriality of F; so the naturality square commutes as required.

5 Finally, we must check the naturality in F; given a natural transformation $\theta\colon F \longrightarrow G$, we show that

$$\begin{array}{ccc} FA & \longrightarrow & [\mathcal{C}^{\text{op}}, \mathbf{Set}](H_A, F) \\ {\scriptstyle \theta_A}\downarrow & & \downarrow{\scriptstyle \theta\circ\_} \\ GA & \longrightarrow & [\mathcal{C}^{\text{op}}, \mathbf{Set}](H_A, G) \end{array}$$

commutes. We have

$$\begin{array}{ccc} x & \longmapsto & \widehat{x} \\ & & \big\downarrow \\ & & \theta \circ \widehat{x} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} x & & \\ \big\downarrow & & \\ \theta_A(x) & \longmapsto & \widehat{\theta_A(x)} \end{array}$$

with respective components

$$\begin{aligned} \mathcal{C}(V, A) &\longrightarrow GA \\ f &\longmapsto \theta_V \circ Ff(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{C}(V, A) &\longrightarrow GA \\ f &\longmapsto Gf \circ \theta_A(x) \end{aligned}$$

But these two are equal by the naturality of θ; so the naturality square commutes as required. □

Dually, for $F \colon \mathcal{C} \longrightarrow \mathbf{Set}$, we have

$$FA \cong [\mathcal{C}, \mathbf{Set}](H^A, F).$$

LECTURE 7 · 25/10/02

THEOREM 2.3.2

The Yoneda embedding is full & faithful.

PROOF

We need to show that $\mathcal{C}(A, B) \xrightarrow{H_\bullet} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, H_B)$ is an isomorphism. By the Yoneda lemma, with $F = H_B$, we have

$$H_B(A) \cong [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, H_B).$$

So we just need to check that $H_\bullet$ is the same isomorphism as that given by the Yoneda lemma; i.e. that $\widehat{f} = H_f$ or $\widehat{H_f} = f$. But

$$\widehat{H_f} = (H_f)_A(1_A) = f.$$

□

Note that this shows that, given $f, g \colon A \longrightarrow B$, then $H_f = H_g \Rightarrow f = g$. Also, given $H_A \xrightarrow{h} H_B$, there exists $f \colon A \longrightarrow B$ such that $H_f = h$.

PROPOSITION 2.3.3

$A \cong B \in \mathcal{C}$ implies $\mathcal{C}(X, A) \cong \mathcal{C}(X, B)$ and $\mathcal{C}(A, X) \cong \mathcal{C}(B, X)$, each isomorphism being natural in X.

PROOF

$H_\bullet$ is full and faithful, so $A \cong B \Leftrightarrow H_A \cong H_B$, so $\mathcal{C}(X, A) \cong \mathcal{C}(X, B)$ naturally in X. Similarly for the dual statement. □

2.4 · *Parametrised representability*

Consider $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Set}$. For all $A \in \mathcal{A}$, we get

$$\begin{aligned} F(\_, A) \colon \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ X &\longmapsto F(X, A). \end{aligned}$$

Suppose each $F(\_, A)$ has a given representation, i.e.

- an object U_A;

- a natural isomorphism $\alpha_A \colon \mathcal{C}(\_, U_A) \longrightarrow F(\_, A)$.

So we have an assignation $A \mapsto U_A$. Can we extend it to a functor? And are the α_A the components of a natural transformation?

PROPOSITION 2.4.1

Given a functor $F \colon \mathcal{C}^{\text{op}} \times A \longrightarrow \mathbf{Set}$ such that each $F(\_, A) \colon \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set}$ has a representation

$$\alpha_A \colon \mathcal{C}(\_, U_A) \longrightarrow F(\_, A),$$

then there is a unique way to extend $A \mapsto U_A$ to a functor $U \colon \mathcal{A} \longrightarrow \mathcal{C}$ such that the α_A are components of a natural transformation $H_\bullet \circ U \longrightarrow F$.

PROOF

First we construct U on morphisms; i.e. given $f \colon A \longrightarrow B$, we seek $Uf \colon U_A \longrightarrow U_B$. In order to satisfy the naturality condition on α, we need

$$\begin{array}{ccc} \mathcal{C}(\_, U_A) & \xrightarrow{\alpha_A} & F(\_, A) \\ \big\downarrow & & \big\downarrow{\scriptstyle F(\_, f)} \\ \mathcal{C}(\_, U_B) & \xrightarrow[\alpha_B]{} & F(\_, B) \end{array}$$

to commute.

Since the horizontal morphisms are isomorphisms, we get a unique morphism on the left $H_{U_A} \longrightarrow H_{U_B}$ making the diagram commute. Now, the Yoneda embedding is full and faithful, so there exists a unique morphism $U_A \longrightarrow U_B$ inducing it. Call this Uf. It only remains to check that U is functorial; it will make α a natural transformation by construction.

1 Check $U(1_A) = 1_{UA}$. Note that $U(1_A)$ is the unique morphism making the naturality square commute, so it suffices to check that 1_{UA} makes the square commute.
We have

$$\begin{array}{ccc} \mathcal{C}(\_, U_A) & \xrightarrow{\alpha_A} & F(\_, A) \\ {\scriptstyle 1_{U_A} \circ \_}\big\downarrow & & \big\downarrow{\scriptstyle F(\_, 1_A)} \\ \mathcal{C}(\_, U_A) & \xrightarrow[\alpha_A]{} & F(\_, A) \end{array}$$

which commutes as required.

2 We check $U(g \circ f) = Ug \circ Uf$ given $A \xrightarrow{f} B \xrightarrow{g} C$. Consider

$$\begin{array}{ccc} \mathcal{C}(\_, U_A) & \xrightarrow{\alpha_A} & F(\_, A) \\ {\scriptstyle H_{Uf}}\big\downarrow & & \big\downarrow{\scriptstyle F(\_, f)} \\ \mathcal{C}(\_, U_B) & \xrightarrow{\alpha_B} & F(\_, B) \\ {\scriptstyle H_{Ug}}\big\downarrow & & \big\downarrow{\scriptstyle F(\_, g)} \\ \mathcal{C}(\_, U_C) & \xrightarrow[\alpha_C]{} & F(\_, C) \end{array}$$

Each square commutes, so the outside commutes. Now, the composite on the RHS is $F(\_, g \circ f)$, and by definition it induces a unique map $H_{U(g\circ f)}$ on the left such that the diagram commutes. So we must have

$$\begin{aligned} H_{U(g\circ f)} &= H_{Ug} \circ H_{Uf} \\ &= H_{Ug\circ Uf}, \end{aligned}$$

by functorality. But the Yoneda embedding is full and faithful, so we have $U(g \circ f) = Ug \circ Uf$ as required. □

DEFINITION 2.4.2

A *Cartesian closed category* is a category $\mathcal{C}$ equipped with:

- a terminal object T;
- binary objects;
- function spaces.

In fact, in the light of the above results on representability, we can also characterise a Cartesian closed category as containing:

- a representation for the functor $F\colon X \mapsto 1$, since $1 \cong \mathcal{C}(X, T)$ for T a terminal object;
- representations for the functors $F_{A,B}\colon X \longrightarrow \mathcal{C}(X, A) \times \mathcal{C}(X, B)$, since $\mathcal{C}(X, A) \times \mathcal{C}(X, B) \cong \mathcal{C}(X, A \times B)$ naturally in X;
- representations for the functors $F_{B,C}\colon X \longrightarrow \mathcal{C}(X\times B, C)$, since $\mathcal{C}(X\times B, C) \cong \mathcal{C}(X, C^B)$ naturally in X.

We can do even better; using the parametrised representability result, we can:

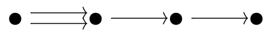
- from the functor $F\colon (X, (A, B)) \mapsto \mathcal{C}(X, A)\times\mathcal{C}(X, B)$, construct the functor $U\colon (A, B) \mapsto A\times B$;
- from the functor $F\colon (X, (B, C)) \mapsto \mathcal{C}(X \times B, C)$ construct the functor $U\colon (B, C) \mapsto C^B$.

LECTURE 8 · 25/10/02

3 · Limits & colimits

3.1 · *Introduction*

Consider any drawable diagram contained within some category $\mathcal{D}$; for example



Then a *limit* over this diagram is a universal cone:

3.1.1 · CONES

A *cone over a diagram* consists of:

- a vertex - an object in $\mathcal{D}$;
- projections - a morphism from the vertex to each object of the diagram,

such that all the resulting triangles commute: